

Commutative hypergroups associated with a hyperfield

Herbert Heyer, Satoshi Kawakami, Tatsuya Tsurii
and Satoe Yamanaka

Abstract

Let H be a commutative hypergroup and L a discrete commutative hypergroup. In the present paper we introduce a commutative hypergroup $\mathcal{K}(H, \varphi, L)$ associated with a hyperfield φ of H based on L . Moreover for the hyperfield φ of a compact commutative hypergroup H of strong type based on a discrete commutative hypergroup L of strong type, we introduce the dual hyperfield $\hat{\varphi}$ of \hat{L} based on \hat{H} and show that $\hat{\mathcal{K}}(H, \varphi, L) \cong \mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H})$.

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1. Introduction

In a previous paper ([HKTY2]) we discussed the hypergroup structure of the space $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$ under the assumptions that H is a compact hypergroup of strong type and H_0 is a closed subhypergroup of H with finite index $[H : H_0]$, \hat{H} and $\widehat{H_0}$ denoting the dual of H and H_0 respectively, and $\mathbb{Z}_q(2)$ signifying the q -deformation of \mathbb{Z}_2 with $0 < q \leq 1$ in the sense of [KTY]. It remained an open problem to investigate the hypergroup structure of the dual $\hat{\mathcal{K}}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$ of $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$ at least under additional assumptions on H and H_0 . It required a different approach to master that problem. The present solution to the problem relies on a generalization of the notion of a hyperfield, which had been successfully applied in the case of finite hypergroups in [HKKK].

Let us emphasize that in the present discussion the dual hypergroup structure of $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$ will be established for commutative hypergroups H without assuming compactness of H and finiteness of the index $[H : H_0]$ of H_0 in H .

The preliminary knowledge on hypergroups needed in the sequel can be taken from the traditional sources [BH] and [J]; some additional references on the structure of hypergroups to be consulted are [HK1], [HK2] and [HKY].

In section 3 the appropriate generalization of the hyperfield method is presented. We are considering hyperfields φ mapping elements ℓ of a countable discrete commutative hypergroup L to compact subhypergroups $H(\ell)$ of a commutative hypergroup H . Then the space $\mathcal{K}(H, \varphi, L)$ is introduced and shown to be a commutative hypergroup (Theorem 3.1). Next we define the hypergroup

$\mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H})$ by the dual field of $\hat{\varphi}$ of φ and obtain that for a compact commutative hypergroup H of strong type the desired dual $\hat{\mathcal{K}}(H, \varphi, L)$ is isomorphic to $\mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H})$ (Theorem 3.5). Natural conditions yield the identification of $\mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H})$ with $\hat{L} \vee \hat{H}$ and with the substitution $S(Q \times L : Q \longrightarrow H)$ introduced by Voit in [V1].

Section 4 contains various examples of the hyperfield method, in particular the extension of Voit's result in [V2] to higher dimensional tori.

In section 5 we drop the assumption of compactness of H and identify $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$ with $\mathcal{K}(\hat{H}, \varphi, \mathbb{Z}_q(2))$ (Theorem 5.2) using the character theory for induced representations developed in [HKY].

2. Preliminaries

For a locally compact space X we shall mainly consider the subspaces $C_c(X)$ and $C_0(X)$ of the space $C(X)$ of continuous functions on X which have compact support or vanish at infinity respectively. By $M(X)$, $M^b(X)$ and $M_c(X)$ we abbreviate the spaces of all (Radon) measures on X , the bounded measures and the measures with compact support on X respectively. Let $M^1(X)$ denote the set of probability measures on X and $M_c^1(X)$ its subset $M^1(X) \cap M_c(X)$. The symbol δ_x stands for the Dirac measures in $x \in X$.

A *hypergroup* $(K, *)$ is a locally compact space K together with a *convolution* $*$ in $M^b(K)$ such that $(M^b(K), *)$ becomes a Banach algebra and that the following properties are fulfilled.

(H1) The mapping

$$(\mu, \nu) \longmapsto \mu * \nu$$

from $M^b(K) \times M^b(K)$ into $M^b(K)$ is continuous with respect to the weak topology in $M^b(K)$.

(H2) For $x, y \in K$ the convolution $\delta_x * \delta_y$ belongs to $M_c^1(K)$.

(H3) There exist a *unit* element $e \in K$ with

$$\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$$

for all $x \in K$, and an *involution*

$$x \longmapsto x^-$$

in K such that

$$\delta_{x^-} * \delta_{y^-} = (\delta_y * \delta_x)^-$$

and

$$e \in \text{supp}(\delta_x * \delta_y) \text{ if and only if } x = y^-$$

whenever $x, y \in K$.

(H4) The mapping

$$(x, y) \longmapsto \text{supp}(\delta_x * \delta_y)$$

from $K \times K$ into the space $\mathcal{C}(K)$ of all compact subsets of K furnished with Michael topology is continuous.

A hypergroup $(K, *)$ is said to be *commutative* if the convolution $*$ is commutative. In this case $(M^b(K), *, -)$ is a commutative Banach $*$ -algebra with identity δ_e . There is an abundance of hypergroups and there are various constructions (polynomial, Sturm-Liouville) as the reader may learn from the pioneering papers on the subject.

Let $(K, *)$ and (L, \circ) be two hypergroups with units e_K and e_L respectively. A continuous mapping $\varphi : K \rightarrow L$ is called a hypergroup *homomorphism* if $\varphi(e_K) = e_L$ and φ is the unique linear, weakly continuous extension from $M^b(K)$ to $M^b(L)$ such that

$$\varphi(\delta_x) = \delta_{\varphi(x)}, \quad \varphi(\delta_x^-) = \varphi(\delta_x)^- \quad \text{and} \quad \varphi(\delta_x * \delta_y) = \varphi(\delta_x) \circ \varphi(\delta_y)$$

whenever $x, y \in K$. If $\varphi : K \rightarrow L$ is also a homeomorphism, it will be called an *isomorphism* from K onto L . An isomorphism from K onto K is called an *automorphism* of K . We denote by $\text{Aut}(K)$ the set of all automorphisms of K . Then $\text{Aut}(K)$ becomes a topological group equipped with the weak topology of $M^b(K)$. We call α an *action* of a locally compact group G on a hypergroup H if α is a continuous homomorphism from G into $\text{Aut}(H)$. Associated with the action α of G on H one can define a semi-direct product hypergroup $K = H \rtimes_\alpha G$.

If the given hypergroup K is commutative, its dual \widehat{K} can be introduced as the set of all bounded continuous functions $\chi \neq 0$ on $M^b(K)$ satisfying

$$\chi(\delta_x * \delta_y) = \chi(\delta_x)\chi(\delta_y) \quad \text{and} \quad \chi(\delta_x^-) = \overline{\chi(\delta_x)}$$

for all $x, y \in K$. This set of characters \widehat{K} of K becomes a locally compact space with respect to the topology of uniform convergence on compact sets, but generally fails to be a hypergroup. If \widehat{K} is a hypergroup, then K is called a strong hypergroup or a hypergroup of strong type. If the dual \widehat{K} of a strong hypergroup K is also strong and $\widehat{\widehat{K}} \cong K$ holds, then K is called a *Pontryagin* hypergroup or a hypergroup of *Pontryagin type*.

3. Hyperfields and hypergroups

Let $H = (H, M^b(H), *, -)$ be a commutative hypergroup with unit h_0 and $L = (L, M^b(L), \bullet, -) = \{\ell_0, \ell_1, \dots, \ell_n, \dots\}$ a countable discrete commutative hypergroup where ℓ_0 is unit of L .

Definition For each $\ell \in L$, let $H(\ell)$ be a compact subhypergroup of H satisfying the following conditions.

- (1) $H(\ell_0) = \{h_0\}$ and $H(\ell^-) = H(\ell)$.
- (2) $[H(\ell_i) * H(\ell_j)] \supset H(\ell_k)$ for $\ell_k \in \text{supp}(\varepsilon_{\ell_i} \bullet \varepsilon_{\ell_j})$, where $[H(\ell_i) * H(\ell_j)]$ is the compact subhypergroup of H generated by $H(\ell_i)$ and $H(\ell_j)$.

Then we call

$$\varphi : L \ni \ell \longmapsto H(\ell) \subset H$$

a *hyperfield* of H based on L .

We denote the normalized Haar measure $\omega_{H(\ell)}$ of $H(\ell)$ by $e(\ell)$ and note that condition (2) implies

$$(3) \quad e(\ell_i) * e(\ell_j) * e(\ell_k) = e(\ell_i) * e(\ell_j) \text{ for } \ell_k \in \text{supp}(\varepsilon_{\ell_i} \bullet \varepsilon_{\ell_j}).$$

Putting

$$\mathcal{K}(H, \varphi, L) := \{(\delta_h * e(\ell)) \otimes \varepsilon_\ell \in M^b(H) \otimes M^b(L) : h \in H, \ell \in L\}$$

one sees that

$$\mathcal{K}(H, \varphi, L) = Q(\ell_0) \cup Q(\ell_1) \cup \dots \cup Q(\ell_n) \cup \dots$$

is a locally compact space, where

$$\begin{aligned} Q(\ell) &:= H/H(\ell) \text{ for all } \ell \in L, \\ Q(\ell_0) &:= H. \end{aligned}$$

Now we shall introduce a convolution \circ and an involution $-$ on $\mathcal{K}(H, \varphi, L)$ as elements of $M^b(H) \otimes M^b(L)$ in order to obtain the following theorem generalizing a result in [HKKK].

Theorem 3.1 Let φ be a hyperfield of a commutative hypergroup H based on a countable discrete commutative hypergroup L . Then $\mathcal{K}(H, \varphi, L)$ is a commutative hypergroup.

Proof The set $\{\delta_h * e(\ell) : h \in H\}$ is a commutative hypergroup isomorphic to the quotient hypergroup

$$Q(\ell) = H/H(\ell) \text{ for all } \ell \in L.$$

We see that

$$M^b(\mathcal{K}(H, \varphi, L)) = \sum_{\ell \in L}^{\oplus} M^b(Q(\ell)).$$

Next we examine the convolution on $\mathcal{K}(H, \varphi, L)$ in detail. For $\ell_i, \ell_j \in L$ we denote the set $\{k : \ell_k \in \text{supp}(\varepsilon_{\ell_i} \bullet \varepsilon_{\ell_j})\}$ by $s(\ell_i, \ell_j)$. Given $h_p, h_q \in H$ and $\ell_i, \ell_j \in L$,

$$\begin{aligned} &((\delta_{h_p} * e(\ell_i)) \otimes \varepsilon_{\ell_i}) \circ ((\delta_{h_q} * e(\ell_j)) \otimes \varepsilon_{\ell_j}) \\ &= (\delta_{h_p} * \delta_{h_q} * e(\ell_i) * e(\ell_j)) \otimes (\varepsilon_{\ell_i} \bullet \varepsilon_{\ell_j}) \end{aligned}$$

$$\begin{aligned}
&= (\delta_{h_p} * \delta_{h_q} * e(\ell_i) * e(\ell_j) * e(\ell_k)) \otimes \left(\sum_{k \in s(\ell_i, \ell_j)} n_{ij}^k \varepsilon_{\ell_k} \right) \\
&= \sum_{k \in s(\ell_i, \ell_j)} n_{ij}^k (\delta_{h_p} * \delta_{h_q} * e(\ell_i) * e(\ell_j) * e(\ell_k)) \otimes \varepsilon_{\ell_k}
\end{aligned}$$

by condition (3) derived from the defining properties of the hyperfield φ . In conclusion the convolution in $\mathcal{K}(H, \varphi, L)$ is well-defined, and its associativity holds.

Now we note that

$$\text{supp}(((\delta_{h_p} * e(\ell_i)) \otimes \varepsilon_{\ell_i}) \circ ((\delta_{h_q} * e(\ell_j)) \otimes \varepsilon_{\ell_j})) = \bigcup_{k \in s(\ell_i, \ell_j)} ((h_p * h_q * [H(\ell_i) * H(\ell_j)]) / H(\ell_k))$$

is compact.

In order to verify the defining property of the involution $^-$ of $\mathcal{K}(H, \varphi, L)$ we compute

$$\begin{aligned}
&((\delta_{h_p} * e(\ell_i)) \otimes \varepsilon_{\ell_i}) \circ ((\delta_{h_q} * e(\ell_j)) \otimes \varepsilon_{\ell_j})^- \\
&= ((\delta_{h_p} * e(\ell_i)) \otimes \varepsilon_{\ell_i}) \circ ((\delta_{h_q}^- * e(\ell_j)) \otimes \varepsilon_{\ell_j}^-) \\
&= \sum_{k \in s(\ell_i, \ell_j^-)} n_{ij}^k (\delta_{h_p} * \delta_{h_q}^- * e(\ell_i) * e(\ell_j) * e(\ell_k)) \otimes \varepsilon_{\ell_k}.
\end{aligned}$$

These equalities imply that

$$((\delta_{h_p} * e(\ell_i)) \otimes \varepsilon_{\ell_i}) = ((\delta_{h_q} * e(\ell_j)) \otimes \varepsilon_{\ell_j})^-$$

holds if and only if

$$(h_0, \ell_0) \in \text{supp}(((\delta_{h_p} * e(\ell_i)) \otimes \varepsilon_{\ell_i}) \circ ((\delta_{h_q} * e(\ell_j)) \otimes \varepsilon_{\ell_j})).$$

The remaining axioms of a hypergroup are easily verified. All together the desired conclusions are established. [Q.E.D.]

Remark If $\varphi(\ell) = H(\ell) = \{h_0\}$ for all $\ell \in L$, then $\mathcal{K}(H, \varphi, L)$ is the direct product hypergroup $H \times L$.

Now let H be a compact commutative hypergroup of strong type such that \hat{H} is a discrete commutative hypergroup and L be a discrete commutative hypergroup of strong type such that \hat{L} is a compact hypergroup. For $\chi \in \hat{H}$ we put

$$Y(\chi) := \{\ell \in L : \chi \in H(\ell)^\perp\}$$

where

$$H(\ell)^\perp := \{\chi \in \hat{H} : \chi(h) = 1 \text{ for all } h \in H(\ell)\}$$

denotes the annihilator of $H(\ell)$ in \hat{H} .

Lemma 3.2 Given $\chi \in \hat{H}$, $Y(\chi)$ is a subhypergroup of L such that $Y(\chi^-) = Y(\chi)$, and $Y(\chi_i) \cap Y(\chi_j) \subset Y(\chi_k)$ for $\chi_k \in \text{supp}(\varepsilon_{\chi_i} * \varepsilon_{\chi_j})$.

Proof First of all we note that

$$H(\ell_i)^\perp \cap H(\ell_j)^\perp = [H(\ell_i) * H(\ell_j)]^\perp \subset H(\ell_k)^\perp$$

for $\ell_k \in \text{supp}(\varepsilon_{\ell_i} \bullet \varepsilon_{\ell_j})$ by the defining property (2) of the hyperfield φ of H based on L .

For $\ell_i, \ell_j \in Y(\chi)$, $\chi \in H(\ell_i)^\perp \cap H(\ell_j)^\perp \subset H(\ell_k)^\perp$ whenever $\ell_k \in \text{supp}(\varepsilon_{\ell_i} \bullet \varepsilon_{\ell_j})$, i.e., $\chi \in H(\ell_k)^\perp$. Then $\text{supp}(\varepsilon_{\ell_i} \bullet \varepsilon_{\ell_j}) \subset Y(\chi)$. For $\ell \in Y(\chi)$ we have $\ell^- \in Y(\chi)$ by property (1) of the definition of the hyperfield φ which states $H(\ell^-) = H(\ell)$ for all $\ell \in L$. Then $Y(\chi)$ appears to be a subhypergroup of L .

For $\ell \in Y(\chi_i) \cap Y(\chi_j)$ we see that $\chi_i \in H(\ell)^\perp$ and $\chi_j \in H(\ell)^\perp$. Since $H(\ell)^\perp$ is a subhypergroup of \hat{H} we get

$$\text{supp}(\varepsilon_{\chi_i} * \varepsilon_{\chi_j}) \subset H(\ell)^\perp,$$

which implies that $\chi_k \in H(\ell)^\perp$ for $\chi_k \in \text{supp}(\varepsilon_{\chi_i} * \varepsilon_{\chi_j})$. Altogether we arrive at the fact that $\ell \in Y(\chi_k)$, i.e., $Y(\chi_i) \cap Y(\chi_j) \subset Y(\chi_k)$. [Q.E.D.]

Denoting $Y(\chi)^\perp$ by $\hat{L}(\chi)$ for $\chi \in \hat{H}$ we easily see that $\hat{L}(\chi)$ is a closed subhypergroup of the compact hypergroup \hat{L} satisfying properties (1) and (2) of the hyperfield φ by Lemma 3.2. This leads to the following.

Lemma 3.3 The mapping

$$\hat{\varphi} : \hat{H} \ni \chi \longmapsto \hat{L}(\chi) \subset \hat{L}$$

is a hyperfield of \hat{L} based on \hat{H} .

Definition The hyperfield $\hat{\varphi}$ is called the *dual* of the hyperfield

$$\varphi : L \ni \ell \longmapsto H(\ell) \subset H$$

of H based on L . We note that the duality $\hat{\hat{\varphi}} = \varphi$ holds if H and L are Pontryagin.

As a consequence of these preparations we obtain a commutative hypergroup

$$\mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}) = \{(\delta_\rho * e(\chi)) \otimes \varepsilon_\chi : \rho \in \hat{L}, \chi \in \hat{H}\},$$

where $e(\chi)$ denotes the normalized Haar measure of $\hat{L}(\chi)$.

The following statements are easily verified.

Lemma 3.4

- (i) For each $\chi \in \hat{H}$ and $\ell \in L$, $\ell \in Y(\chi)$ if and only if $\chi \in H(\ell)^\perp$

(ii) For each $\chi \in \hat{H}$ and the Haar measure $e(\ell)$ of $H(\ell)$

$$\chi(e(\ell)) = \begin{cases} 1 & \text{if } \chi \in H(\ell)^\perp \\ 0 & \text{otherwise} \end{cases}$$

(iii) For each $\ell \in L$ and the Haar measure $e(\chi)$ of $\hat{L}(\chi)$

$$e(\chi)(\ell) = \begin{cases} 1 & \text{if } \ell \in Y(\chi) \\ 0 & \text{otherwise} \end{cases}$$

(iv) For each $\chi \in \hat{H}$ and $\ell \in L$, $\chi(e(\ell)) = e(\chi)(\ell)$.

Now, we arrive at the dual version of the statement of Theorem 3.1.

Theorem 3.5 Let φ be a hyperfield of a compact commutative hypergroup H of strong type based on a discrete commutative hypergroup L of strong type. Then

$$\hat{\mathcal{K}}(H, \varphi, L) \cong \mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}).$$

If H and L are Pontryagin hypergroups, then $\mathcal{K}(H, \varphi, L)$ is also Pontryagin. Moreover the sequence

$$1 \longrightarrow H \longrightarrow \mathcal{K}(H, \varphi, L) \longrightarrow L \longrightarrow 1$$

is exact and the dual sequence

$$1 \longrightarrow \hat{L} \longrightarrow \mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}) \longrightarrow \hat{H} \longrightarrow 1$$

is exact as well. In particular $\mathcal{K}(H, \varphi, L)$ and $\mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H})$ are extension hypergroups of L by H and \hat{H} by \hat{L} respectively.

Proof Clearly

$$\hat{\mathcal{K}}(H, \varphi, L) \supset \mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}).$$

It remains to be shown that

$$\hat{\mathcal{K}}(H, \varphi, L) \subset \mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}).$$

Let τ be a character of $\mathcal{K}(H, \varphi, L)$. Then there exists $\chi \in \hat{H}$ such that

$$\tau((\delta_h * e(\ell)) \otimes \varepsilon_\ell) = \chi(h)\chi(e(\ell))\rho(\ell) = \rho(\ell)e(\chi)(\ell)\chi(h) = (\delta_\rho * e(\chi)) \otimes \varepsilon_\chi(\ell, h)$$

for some $\rho \in \hat{L}$ by Lemma 3.4. Consequently

$$\tau = (\delta_\rho * e(\chi)) \otimes \varepsilon_\chi \in \mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}).$$

The assertion concerning the Pontryagin property follows from the fact that the dual $\hat{\varphi}$ of the dual hyperfield φ is φ .

Now let $e(H)$ denote the normalized Haar measure of the compact hypergroup H . Then

$$\begin{aligned} Q &:= \{(e(H) \otimes \varepsilon_{\ell_0}) \circ (\delta_h \otimes \varepsilon_{\ell_i}) : h \in H, \ell_i \in L\} \\ &= \{e(H) \otimes \varepsilon_{\ell_i} : \ell_i \in L\} \end{aligned}$$

is the quotient hypergroup $\mathcal{K}(H, \varphi, L)/H$ isomorphic to L . This means that $\mathcal{K}(H, \varphi, L)$ is an extension hypergroup of L by H . [Q.E.D.]

Remark

(1) If $\varphi(\ell_0) = \{h_0\}$ and $\varphi(\ell) = H$ for all $\ell \in L$ such that $\ell \neq \ell_0$, then $\mathcal{K}(H, \varphi, L)$ is the hypergroup join $H \vee L$. Moreover, if H and L are strong, then

$$\mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}) = \hat{L} \vee \hat{H}.$$

(2) Let $Q := H/H_0$ for a closed subhypergroup H_0 of H . If $\varphi(\ell_0) = \{h_0\}$ and $\varphi(\ell) = H_0$ for all $\ell \in L$, $\ell \neq \ell_0$, then

$$\mathcal{K}(H, \varphi, L) = S(Q \times L : Q \longrightarrow H),$$

where the latter symbol denotes the substitution hypergroup obtained by substituting Q in $Q \times L$ by H , in the sense of Voit [V1]. We note that H_0 is not assumed to be open which means that our definition is a generalization of Voit's substitution.

4. Examples of hyperfields

Let $\mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ be the hypergroup of order two, where the convolution structure is given by

$$\varepsilon_{\ell_1} \bullet \varepsilon_{\ell_1} = q\varepsilon_{\ell_0} + (1 - q)\varepsilon_{\ell_1}$$

for $0 < q \leq 1$.

Example 4.1 If $H = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $L = \mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ and $\varphi : \mathbb{Z}_q(2) \ni \ell \mapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{1\}$, $\varphi(\ell_1) = H(\ell_1) = C_n := \{z \in \mathbb{T} : z^n = 1\}$ ($n \in \mathbb{N}$), then

$$\mathcal{K}(H, \varphi, L) = \mathcal{K}(\mathbb{T}, \varphi, \mathbb{Z}_q(2)) = \mathbb{T} \cup \mathbb{T}$$

is a commutative hypergroup which coincides with Voit's commutative hypergroup on the two tori $\mathbb{T} \cup \mathbb{T}$ ([V2]).

This means that $\mathcal{K}(\mathbb{T}, \varphi, \mathbb{Z}_q(2))$ determines the commutative hypergroup structure on $\mathbb{T} \cup \mathbb{T}$ with parameter (n, q) , $n \in \mathbb{N}$, $0 < q \leq 1$.

Obviously $\mathcal{K}(\mathbb{T}, \varphi, \mathbb{Z}_q(2))$ is Pontryagin and

$$\hat{\mathcal{K}}(\mathbb{T}, \varphi, \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, \mathbb{Z}),$$

where the dual field $\hat{\varphi}$ of φ is given as

$$\hat{\varphi} : \mathbb{Z} \ni k \longmapsto \hat{\varphi}(k) \subset \mathbb{Z}_q(2)$$

with

$$\hat{\varphi}(k) = \begin{cases} \{\ell_0\} & \text{for } k \in n\mathbb{Z} \\ \mathbb{Z}_q(2) & \text{otherwise.} \end{cases}$$

Example 4.2 If $H = \mathbb{T} \times \mathbb{T} = \mathbb{T}^2$, $L = \mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ and $\varphi : \mathbb{Z}_q(2) \ni \ell \longmapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{(1, 1)\}$, $\varphi(\ell_1) = H(\ell_1) = C_n \times C_m$ ($n, m \in \mathbb{N}$), then

$$\mathcal{K}(H, \varphi, L) = \mathcal{K}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2)) = \mathbb{T}^2 \cup \mathbb{T}^2$$

is a commutative hypergroup.

Obviously $\mathcal{K}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2))$ is Pontryagin and

$$\hat{\mathcal{K}}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, \mathbb{Z}^2),$$

where the dual field $\hat{\varphi}$ of φ is given as

$$\hat{\varphi} : \mathbb{Z} \times \mathbb{Z} \ni k \longmapsto \hat{\varphi}(k) \subset \mathbb{Z}_q(2)$$

with

$$\hat{\varphi}(k) = \begin{cases} \{\ell_0\} & \text{for } k \in n\mathbb{Z} \times m\mathbb{Z} \\ \mathbb{Z}_q(2) & \text{otherwise.} \end{cases}$$

Example 4.3 If $H = \mathbb{T}$, $L = \mathbb{Z}_q(3) = \{\ell_0, \ell_1, \ell_2\}$ (see [KTY]) and $\varphi : \mathbb{Z}_q(3) \ni \ell \longmapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{1\}$, $\varphi(\ell_1) = H(\ell_1) = C_n$ and $\varphi(\ell_2) = H(\ell_2) = C_n$, then

$$\mathcal{K}(H, \varphi, L) = \mathcal{K}(\mathbb{T}, \varphi, \mathbb{Z}_q(3)) = \mathbb{T} \cup \mathbb{T} \cup \mathbb{T}$$

is a commutative hypergroup.

Obviously $\mathcal{K}(\mathbb{T}, \varphi, \mathbb{Z}_q(3))$ is Pontryagin and

$$\hat{\mathcal{K}}(\mathbb{T}, \varphi, \mathbb{Z}_q(3)) = \mathcal{K}(\mathbb{Z}_q(3), \hat{\varphi}, \mathbb{Z}),$$

where the dual field $\hat{\varphi}$ of φ is given as

$$\hat{\varphi} : \mathbb{Z} \ni k \longmapsto \hat{\varphi}(k) \subset \mathbb{Z}_q(3)$$

with

$$\hat{\varphi}(k) = \begin{cases} \{\ell_0\} & \text{for } k \in n\mathbb{Z} \\ \mathbb{Z}_q(3) & \text{otherwise.} \end{cases}$$

Example 4.4 If $H = \mathbb{T} \times \mathbb{T} = \mathbb{T}^2$, $L = \mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ and $\varphi : \mathbb{Z}_q(2) \ni \ell \mapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{(1, 1)\}$, $\varphi(\ell_1) = H(\ell_1) = C_n \times \mathbb{T}$ ($n \in \mathbb{N}$), then

$$\mathcal{K}(H, \varphi, L) = \mathcal{K}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2)) = \mathbb{T}^2 \cup \mathbb{T}$$

is a commutative hypergroup.

Obviously $\mathcal{K}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2))$ is Pontryagin and

$$\hat{\mathcal{K}}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, \mathbb{Z}^2),$$

where the dual field $\hat{\varphi}$ of φ is given as

$$\hat{\varphi} : \mathbb{Z} \times \mathbb{Z} \ni k \mapsto \hat{\varphi}(k) \subset \mathbb{Z}_q(2)$$

with

$$\hat{\varphi}(k) = \begin{cases} \{\ell_0\} & \text{for } k \in n\mathbb{Z} \times \{0\} \\ \mathbb{Z}_q(2) & \text{otherwise.} \end{cases}$$

Example 4.5 Let $H = \mathcal{K}^\alpha(\mathbb{T}) = [-1, 1]$, $L = \mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ and $\varphi : \mathbb{Z}_q(2) \ni \ell \mapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{1\}$, $\varphi(\ell_1) = H(\ell_1) = \mathcal{K}^\alpha(C_n)$ ($n \in \mathbb{N}$), where $\mathcal{K}^\alpha(\mathbb{T}) = [-1, 1]$ is the orbital hypergroup defined by the action of $\mathbb{Z}_2 = \{e, g\}$ ($g^2 = e$) such that $\alpha_g(z) = \bar{z}$ on \mathbb{T} . Then

$$\mathcal{K}(H, \varphi, L) = \mathcal{K}(\mathcal{K}^\alpha(\mathbb{T}), \varphi, \mathbb{Z}_q(2)) = [-1, 1] \cup [-1, 1]$$

is a commutative hypergroup.

Obviously $\mathcal{K}(\mathcal{K}^\alpha(\mathbb{T}), \varphi, \mathbb{Z}_q(2))$ is Pontryagin and

$$\hat{\mathcal{K}}(\mathcal{K}^\alpha(\mathbb{T}), \varphi, \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, \mathcal{K}^\alpha(\mathbb{Z})),$$

where $\mathcal{K}^\alpha(\mathbb{Z}) = \{0, 1, 2, \dots, n, \dots\}$ is also the orbital hypergroup by the action of $\mathbb{Z}_2 = \{e, g\}$ ($g^2 = e$) such that $\alpha_g(n) = -n$ on \mathbb{Z} and the dual field $\hat{\varphi}$ of φ is given as

$$\hat{\varphi} : \mathcal{K}^\alpha(\mathbb{Z}) \ni k \mapsto \hat{\varphi}(k) \subset \mathbb{Z}_q(2)$$

with

$$\hat{\varphi}(k) = \begin{cases} \{\ell_0\} & \text{for } k \in \mathcal{K}^\alpha(n\mathbb{Z}) \\ \mathbb{Z}_q(2) & \text{otherwise.} \end{cases}$$

Example 4.6 Let A be a commutative strong hypergroup and C a compact strong hypergroup. If $H = A \times C$, $L = \mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ and $\varphi : \mathbb{Z}_q(2) \ni \ell \mapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{h_0\}$, $\varphi(\ell_1) = H(\ell_1) = C$, then

$$\mathcal{K}(H, \varphi, L) = \mathcal{K}(A \times C, \varphi, \mathbb{Z}_q(2)) = (A \times C) \cup A$$

is the commutative hypergroup $A \times (C \vee \mathbb{Z}_q(2))$ of strong type and

$$\widehat{\mathcal{K}}(A \times C, \varphi, \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, \hat{A} \times \hat{C}) = \hat{A} \times (\mathbb{Z}_q(2) \vee \hat{C}).$$

In fact, the dual field $\hat{\varphi}$ of φ is given as

$$\hat{\varphi} : \hat{A} \times \hat{C} \ni (\chi, \rho) \longmapsto \hat{\varphi}(\chi, \rho) \subset \mathbb{Z}_q(2)$$

with

$$\hat{\varphi}(\chi, \rho) = \begin{cases} \{\ell_0\} & \text{for } (\chi, \rho_0) \in \hat{A} \times \hat{C} \\ \mathbb{Z}_q(2) & \text{otherwise,} \end{cases}$$

where ρ_0 is unit of \hat{C} .

Example 4.7 Let A be a commutative hypergroup and C a compact hypergroup. If $H = A \times C$, $L = \mathbb{Z}_q(3) = \{\ell_0, \ell_1, \ell_2\}$ and $\varphi : \mathbb{Z}_q(3) \ni \ell \longmapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{h_0\}$, $\varphi(\ell_1) = H(\ell_1) = C_0$, $\varphi(\ell_2) = H(\ell_2) = C_0$, where C_0 is a closed subhypergroup of C , then

$$\mathcal{K}(H, \varphi, L) = \mathcal{K}(A \times C, \varphi, \mathbb{Z}_q(3)) = (A \times C) \cup (A \times Q) \cup (A \times Q)$$

is the commutative hypergroup $A \times (S(Q \times \mathbb{Z}_q(3) : Q \rightarrow C))$, where $Q = C/C_0$.

5. Applications of the theorems

We assume H to be a (not necessarily compact) commutative hypergroup of strong type such that \hat{H} is a commutative hypergroup with unit character χ_0 . Let H_0 be a closed subhypergroup of strong type of H , where the annihilator H_0^\perp in \hat{H} is a compact subhypergroup of \hat{H} . For $\tau \in \widehat{H_0}$ we consider the set

$$A(\tau) = \{\chi \in \hat{H} : \text{res}_{H_0}^H \chi = \tau\}.$$

As usual $\omega_{H_0^\perp}$ denotes the normalized Haar measure of H_0^\perp . Then there exists a unique H_0^\perp -invariant probability measure $\mu_{A(\tau)}$ which is given by

$$\mu_{A(\tau)} = ch(\tilde{\tau}) \cdot \omega_{H_0^\perp}.$$

for some $\tilde{\tau} \in A(\tau)$.

Definition (see [HKY])

(i) For $\tau \in \widehat{H_0}$ the character of τ induced from H_0 to H is defined by

$$\text{ind}_{H_0}^H ch(\tau) := \mu_{A(\tau)}.$$

(ii) For $\tau_i, \tau_j \in \widehat{H_0}$, $ch(\tau_i) \cdot ch(\tau_j)$ is decomposed on $\widehat{H_0}$ in the form

$$ch(\tau_i) \cdot ch(\tau_j) = \int_C ch(\tau) \nu(d\tau),$$

where ν is a probability measure on $\widehat{H_0}$ and

$$C := \text{supp}(\nu) = \text{supp}(ch(\tau_i) \cdot ch(\tau_j))$$

is compact.

Then we introduce

$$\text{ind}_{H_0}^H(ch(\tau_i) \cdot ch(\tau_j)) := \int_C \text{ind}_{H_0}^H ch(\tau) \nu(d\tau).$$

The subsequent simple facts play an essential role in the upcoming discussion.

Lemma 5.1 (see [HKY])

(i) For $\tau \in \widehat{H_0}$

$$\text{res}_{H_0}^H(\text{ind}_{H_0}^H ch(\tau)) = ch(\tau).$$

(ii) For $\pi_i, \pi_j \in \hat{H}$

$$\text{res}_{H_0}^H(ch(\pi_i) \cdot ch(\pi_j)) = (\text{res}_{H_0}^H ch(\pi_i)) \cdot (\text{res}_{H_0}^H ch(\pi_j)).$$

(iii) For $\pi \in \hat{H}$ and $\tau_i, \tau_j \in \widehat{H_0}$

$$\text{ind}_{H_0}^H((\text{res}_{H_0}^H ch(\pi)) \cdot ch(\tau_i) \cdot ch(\tau_j)) = ch(\pi) \cdot \text{ind}_{H_0}^H(ch(\tau_i) \cdot ch(\tau_j))$$

(iv) For $\tau_i, \tau_j \in \widehat{H_0}$

$$\text{res}_{H_0}^H(\text{ind}_{H_0}^H(ch(\tau_i) \cdot ch(\tau_j))) = ch(\tau_i)ch(\tau_j).$$

Proof (i) and (ii) are clear.

(iii) It is easy to check that

$$\text{ind}_{H_0}^H(ch(\tau_i) \cdot ch(\tau_j)) = ch(\tilde{\tau}_i) \cdot ch(\tilde{\tau}_j) \cdot \omega_{H_0^\perp}$$

for $\tilde{\tau}_i \in A(\tau_i)$ and $\tilde{\tau}_j \in A(\tau_j)$. Then we see that

$$ch(\pi) \cdot \text{ind}_{H_0}^H(ch(\tau_i) \cdot ch(\tau_j)) = ch(\pi) \cdot ch(\tilde{\tau}_i) \cdot ch(\tilde{\tau}_j) \cdot \omega_{H_0^\perp}$$

and

$$\text{ind}_{H_0}^H((\text{res}_{H_0}^H ch(\pi)) \cdot ch(\tau_i) \cdot ch(\tau_j)) = ch(\pi) \cdot ch(\tilde{\tau}_i) \cdot ch(\tilde{\tau}_j) \cdot \omega_{H_0^\perp}.$$

(iv) For $\tau_i, \tau_j \in \widehat{H_0}$

$$\text{res}_{H_0}^H(\text{ind}_{H_0}^H(ch(\tau_i) \cdot ch(\tau_j))) = \text{res}_{H_0}^H \left(\int_C \text{ind}_{H_0}^H ch(\tau) \nu(d\tau) \right)$$

$$\begin{aligned}
&= \int_C \text{res}_{H_0}^H(\text{ind}_{H_0}^H ch(\tau)) \nu(d\tau) \\
&= \int_C ch(\tau) \nu(d\tau) \\
&= ch(\tau_i) \cdot ch(\tau_j).
\end{aligned}$$

Remark A pair (H, H_0) for a commutative hypergroup of strong type is always an admissible hypergroup pair in the sense of [HKTY2] by Lemma 5.1.

Definition (see [HKTY2]) On the space

$$\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)) := \{(ch(\pi), \circ), (ch(\tau), \bullet) : \pi \in \hat{H}, \tau \in \widehat{H_0}\}$$

we define the convolution $*$ by the following properties:

1. $(ch(\pi_i), \circ) * (ch(\pi_j), \circ) := (ch(\pi_i) \cdot ch(\pi_j), \circ),$
2. $(ch(\pi), \circ) * (ch(\tau), \bullet) := ((\text{res}_{H_0}^H ch(\pi)) \cdot ch(\tau), \bullet),$
3. $(ch(\tau), \bullet) * (ch(\pi), \circ) := (ch(\tau) \cdot (\text{res}_{H_0}^H ch(\pi)), \bullet),$
4. $(ch(\tau_i), \bullet) * (ch(\tau_j), \bullet) := q(\text{ind}_{H_0}^H(ch(\tau_i) \cdot ch(\tau_j)), \circ) + (1-q)(ch(\tau_i) \cdot ch(\tau_j), \bullet).$

Definition Given $\mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ ($0 < q \leq 1$) we introduce the set $\mathcal{K}(\hat{H}, \varphi, \mathbb{Z}_q(2))$ via the hyperfield $\varphi : \mathbb{Z}_q(2) \ni \ell \mapsto \hat{\varphi}(\ell) \subset \hat{H}$ given by

$$\varphi(\ell) = \begin{cases} \{\chi_0\} & \text{if } \ell = \ell_0 \\ H_0^\perp & \text{if } \ell = \ell_1 \end{cases}$$

as in section 3.

Then we have

Theorem 5.2 Let H be a commutative hypergroup of strong type and H_0 a closed subhypergroup of H such that H_0^\perp is compact in \hat{H} . Then $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$ is a commutative hypergroup and

$$\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)) \cong \mathcal{K}(\hat{H}, \varphi, \mathbb{Z}_q(2)).$$

Proof In order to show that $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$ is a hypergroup we should check the following associativity relations. For $\pi_i, \pi_j, \pi_k, \pi \in \hat{H}$ and $\tau_i, \tau_j, \tau_k, \tau \in \widehat{H_0}$

$$(A1) \quad ((ch(\pi_i), \circ) * (ch(\pi_j), \circ)) * (ch(\pi_k), \circ) = (ch(\pi_i), \circ) * ((ch(\pi_j), \circ) * (ch(\pi_k), \circ)).$$

$$\begin{aligned}
(A2) \quad & ((ch(\pi_i), \circ) * (ch(\pi_j), \circ)) * (ch(\tau), \bullet) = (ch(\pi_i), \circ) * ((ch(\pi_j), \circ) * (ch(\tau), \bullet)). \\
(A3) \quad & ((ch(\pi), \circ) * (ch(\tau_i), \bullet)) * (ch(\tau_j), \bullet) = (ch(\pi), \circ) * ((ch(\tau_i), \bullet) * (ch(\tau_j), \bullet)). \\
(A4) \quad & ((ch(\tau_i), \bullet) * (ch(\tau_j), \bullet)) * (ch(\tau_k), \bullet) = (ch(\tau_i), \bullet) * ((ch(\tau_j), \bullet) * (ch(\tau_k), \bullet)).
\end{aligned}$$

However these relations are shown in a similar way to the proof of Proposition 3.6 in our paper [HKTY2] combined with the above Lemma 5.1 so that we omit the details. It is easy to check the remaining axioms of a hypergroup for $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$. The desired conclusion is obtained.

Next we introduce an isomorphism $\psi : \mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)) \longrightarrow \mathcal{K}(\hat{H}, \varphi, \mathbb{Z}_q(2))$ by

$$\psi((ch(\pi), \circ)) = ch(\pi) \otimes \varepsilon_{\ell_0}, \quad \psi((ch(\tau), \bullet)) = (ch(\tilde{\tau}) \cdot \omega_{H_0^\perp}) \otimes \varepsilon_{\ell_1}.$$

It is easy to see that ψ is bijective. We only show that ψ is homomorphic.

$$\begin{aligned}
1. \quad & \psi((ch(\pi_i), \circ) * (ch(\pi_j), \circ)) = \psi((ch(\pi_i) \cdot ch(\pi_j), \circ)) \\
& = (ch(\pi_i) \cdot ch(\pi_j)) \otimes \varepsilon_{\ell_0} \\
& = (ch(\pi_i) \otimes \varepsilon_{\ell_0}) \circ (ch(\pi_j) \otimes \varepsilon_{\ell_0}) \\
& = \psi((ch(\pi_i), \circ)) \circ \psi((ch(\pi_j), \circ)),
\end{aligned}$$

$$\begin{aligned}
2. \quad & \psi((ch(\pi), \circ) * (ch(\tau), \bullet)) = \psi(((\text{res}_{H_0}^H ch(\pi)) \cdot ch(\tau), \bullet)) \\
& = (ch(\pi) \cdot ch(\tilde{\tau}) \cdot \omega_{H_0^\perp}) \otimes \varepsilon_{\ell_1} \\
& = (ch(\pi) \otimes \varepsilon_{\ell_0}) \circ (ch(\tilde{\tau}) \cdot \omega_{H_0^\perp} \otimes \varepsilon_{\ell_1}) \\
& = \psi((ch(\pi), \circ)) \circ \psi((ch(\tau), \bullet))
\end{aligned}$$

$$3. \quad \psi((ch(\tau), \bullet) * (ch(\pi), \circ)) = \psi((ch(\tau), \bullet)) \circ \psi((ch(\pi), \circ))$$

is obtained similarly.

$$\begin{aligned}
4. \quad & \psi((ch(\tau_i), \bullet) * (ch(\tau_j), \bullet)) \\
& = \psi(q(\text{ind}_{H_0}^H(ch(\tau_i) \cdot ch(\tau_j)), \circ) + (1-q)(ch(\tau_i) \cdot ch(\tau_j), \bullet)) \\
& = q\psi((\text{ind}_{H_0}^H(ch(\tau_i) \cdot ch(\tau_j)), \circ)) + (1-q)\psi((ch(\tau_i) \cdot ch(\tau_j), \bullet)) \\
& = q((ch(\tilde{\tau}_i) \cdot ch(\tilde{\tau}_j) \cdot \omega_{H_0^\perp}) \otimes \varepsilon_{\ell_0}) + (1-q)((ch(\tilde{\tau}_i) \cdot ch(\tilde{\tau}_j) \cdot \omega_{H_0^\perp}) \otimes \varepsilon_{\ell_1}) \\
& = (ch(\tilde{\tau}_i) \cdot ch(\tilde{\tau}_j) \cdot \omega_{H_0^\perp}) \otimes (q\varepsilon_{\ell_0} + (1-q)\varepsilon_{\ell_1}) \\
& = (ch(\tilde{\tau}_i) \cdot ch(\tilde{\tau}_j) \cdot \omega_{H_0^\perp}) \otimes (\varepsilon_{\ell_1} \bullet \varepsilon_{\ell_1}) \\
& = ((ch(\tilde{\tau}_i) \cdot \omega_{H_0^\perp}) \otimes \varepsilon_{\ell_1}) \circ ((ch(\tilde{\tau}_j) \cdot \omega_{H_0^\perp}) \otimes \varepsilon_{\ell_1}) \\
& = \psi((ch(\tau_i), \bullet)) \circ \psi((ch(\tau_j), \bullet)). \tag{Q.E.D.}
\end{aligned}$$

Now let H be a discrete commutative hypergroup of Pontryagin type and H_0 a closed subhypergroup of H . Since \hat{H} is a compact hypergroup, H_0^\perp is a compact subhypergroup. Then $\mathcal{K}(\hat{H}, \varphi, \mathbb{Z}_q(2))$ is defined via the hyperfield φ given by

$$\varphi(\ell) = \begin{cases} \{\chi_0\} & \text{if } \ell = \ell_0 \\ H_0^\perp & \text{if } \ell = \ell_1. \end{cases}$$

The dual field $\hat{\varphi}$ of φ is the field

$$\hat{\varphi} : H \ni h \mapsto \hat{\varphi}(h) \subset \mathbb{Z}_q(2)$$

with

$$\hat{\varphi}(h) = \begin{cases} \{\ell_0\} & \text{if } h \in H_0 \\ \mathbb{Z}_q(2) & \text{otherwise.} \end{cases}$$

Applying Theorem 3.5 together with Theorem 5.2 we obtain

Theorem 5.3 Let H be a discrete commutative hypergroup of Pontryagin type and H_0 a closed subhypergroup of H such that H_0^\perp is compact in \hat{H} . Then

$$\hat{\mathcal{K}}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)) \cong \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, H).$$

6. Examples of hypergroup duals of $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$

Example 6.1 If $H = \mathbb{Z}$ and $H_0 = n\mathbb{Z}$ ($n \in \mathbb{N}$), then

$$\mathcal{K}(\hat{\mathbb{Z}} \cup \widehat{n\mathbb{Z}}, \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{T}, \varphi, \mathbb{Z}_q(2)) = \mathbb{T} \cup \mathbb{T} \quad (\text{in Example 4.1})$$

and

$$\hat{\mathcal{K}}(\hat{\mathbb{Z}} \cup \widehat{n\mathbb{Z}}, \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, \mathbb{Z}) \quad (\text{in Example 4.1}).$$

Example 6.2 If $H = \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $H_0 = n\mathbb{Z} \times m\mathbb{Z}$ ($n, m \in \mathbb{N}$), then

$$\mathcal{K}(\widehat{\mathbb{Z}^2} \cup (\widehat{n\mathbb{Z} \times m\mathbb{Z}}), \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2)) = \mathbb{T}^2 \cup \mathbb{T}^2 \quad (\text{in Example 4.2})$$

and

$$\hat{\mathcal{K}}(\widehat{\mathbb{Z}^2} \cup (\widehat{n\mathbb{Z} \times m\mathbb{Z}}), \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, \mathbb{Z}^2) \quad (\text{in Example 4.2}).$$

Example 6.3 If $H = \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $H_0 = n\mathbb{Z} \times \{0\} \cong n\mathbb{Z}$ ($n \in \mathbb{N}$), then

$$\mathcal{K}(\widehat{\mathbb{Z}^2} \cup (\widehat{n\mathbb{Z}}), \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2)) = \mathbb{T}^2 \cup \mathbb{T} \quad (\text{in Example 4.4})$$

and

$$\hat{\mathcal{K}}(\widehat{\mathbb{Z}^2} \cup (\widehat{n\mathbb{Z}}), \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, \mathbb{Z}^2) \quad (\text{in Example 4.4}).$$

Example 6.4 If $H = \mathcal{K}^\alpha(\mathbb{Z})$ and $H_0 = \mathcal{K}^\alpha(n\mathbb{Z})$ ($n \in \mathbb{N}$), then

$$\mathcal{K}(\widehat{\mathcal{K}^\alpha(\mathbb{Z}) \cup (\mathcal{K}^\alpha(n\mathbb{Z}))}, \mathbb{Z}_q(2)) = \mathcal{K}(\mathcal{K}^\alpha(\mathbb{T}), \varphi, \mathbb{Z}_q(2)) \quad (\text{in Example 4.5})$$

and

$$\widehat{\mathcal{K}(\mathcal{K}^\alpha(\mathbb{Z}) \cup (\mathcal{K}^\alpha(n\mathbb{Z})), \mathbb{Z}_q(2))} = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, \mathcal{K}^\alpha(\mathbb{Z})) \quad (\text{in Example 4.5}).$$

Example 6.5 Let B be a commutative Pontryagin hypergroup and D a discrete commutative Pontryagin hypergroup. If $H = B \times D$ and $H_0 = B$, then

$$\mathcal{K}(\widehat{B \times D \cup \hat{B}}, \mathbb{Z}_q(2)) = \mathcal{K}(\hat{B} \times \hat{D}, \varphi, \mathbb{Z}_q(2)) = \hat{B} \times (\hat{D} \vee \mathbb{Z}_q(2)) \quad (\text{in Example 4.6})$$

and

$$\widehat{\mathcal{K}(\widehat{B \times D \cup \hat{B}}, \mathbb{Z}_q(2))} = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, B \times D) = B \times (\mathbb{Z}_q(2) \vee D) \quad (\text{in Example 4.6}).$$

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Addresses

Herbert Heyer : Universität Tübingen
 Mathematisches Institut
 Auf der Morgenstelle 10
 72076, Tübingen
 Germany
 e-mail : herbert.heyer@uni-tuebingen.de

Satoshi Kawakami : Nara University of Education
 Department of Mathematics
 Takabatake-cho
 Nara, 630-8528
 Japan
 e-mail : kawakami@nara-edu.ac.jp

Tatsuya Tsurii : Osaka Prefecture University
 Graduate School of Science
 1-1 Gakuen-cho, Nakaku, Sakai
 Osaka, 599-8531
 Japan
 e-mail : dw301003@edu.osakafu-u.ac.jp

Satoe Yamanaka : Nara Women's University
 Faculty of Science
 Kita-uoya-higashi-machi,
 Nara, 630-8506
 Japan
 e-mail : s.yamanaka516@gmail.com